Lecture 14
14.5 Directional derivatives and the gradient

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Things to note

Upcoming dates:
Wednesday, February 27: Quiz 7
Monday, March 4: Quiz 8 and WF drop date (see grade calculation sheet on Blackboard)
Wednesday, March 6: Review
Friday, March 8: Exam 2

Last class

Theorem

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Theorem

If $w = f(x, y)$ is differentiable and if $x = x(s, t)$, $y = y(s, t)$ are differentiable functions of s and t, then the composite $w = f(x(s, t), y(s, t))$ is a differentiable function of s and t and

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

Chain rule example

Example

Find $\frac{\partial w}{\partial r}$ *and* $\frac{\partial w}{\partial s}$ *in terms of* r *and* s *if* $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln(s)$, $z = 2r$

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We have

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (1)(\frac{1}{s}) + (2)(2r) + (2z)(2) = \frac{1}{s} + 12r$$

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and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)(-\frac{r}{s^2}) + (2)(\frac{1}{s}) + (2z)(0) = \frac{-r + 2s}{s^2}$$

14.5 Directional derivatives and the gradient

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Definition

The derivative of f at (a, b) in the direction of the unit vector $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ is the number

$$\left(\frac{df}{dt}\right)_{\vec{\mathbf{u}},(a,b)} = (D_{\vec{\mathbf{u}}} f)(a, b) = \lim_{t \to 0} \frac{f(a + u_1 t, b + u_2 t) - f(a, b)}{t}$$

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We won't use this definition for calculations, but observe that if we take $\vec{\mathbf{u}} = \langle 1, 0 \rangle$ and $\vec{\mathbf{u}} = \langle 0, 1 \rangle$, we get the formal definitions for the partial derivatives from 14.3.

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$$(D_{\vec{\mathbf{u}}} f)(a, b) = \left(\frac{df}{dt} \right)_{\vec{\mathbf{u}},(a,b)}$$

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$$= \left[\frac{\partial f}{\partial x}(a, b)\right] \cdot u_1 + \left[\frac{\partial f}{\partial y}(a, b)\right] \cdot u_2$$

$$= \left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle \cdot \langle u_1, u_2 \rangle$$

We refer to the vector on the left as *the gradient*.

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In the vein of Problem 1 on exam 1, ∇f is *a vector-valued function*. Because of this, $\nabla f(a, b)$ is *a vector (with numbers in it)*.

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Thus

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The directional derivative is a dot product of two vectors, i.e., a *number*.

Example

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Find the derivative of $f(x,y) = xe^y + \cos(xy)$ *at* $(2,0)$ *in the direction of* $\vec{\mathbf{v}} = 3\vec{\mathbf{i}} - 4\vec{\mathbf{j}}$.

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First we need to find $\vec{\mathbf{u}}$, a unit vector in the direction of $\vec{\mathbf{v}}$.

$$\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{\langle 3, -4\rangle}{\sqrt{3^2+4^2}} = \frac{\langle 3, -4\rangle}{5} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle.$$

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$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \langle e^y - y\sin(xy), xe^y - x\sin(xy)\rangle.$$

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Last we evaluate the gradient at $(2,0)$ and dot it with $\vec{\mathbf{u}}$.

$$(D_{\vec{\mathbf{u}}}f)(a,b) = \langle 1-0, 2-0\rangle \cdot \langle \frac{3}{5}, -\frac{4}{5}\rangle = \frac{3}{5} - \frac{8}{5} = -1.$$